

## HYPERBOLIC SPINOR DARBOUX EQUATIONS OF SPACELIKE CURVES IN MINKOWSKI 3-SPACE

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ABSTRACT. In this paper, we study on spinors with two hyperbolic components. Firstly, we express the hyperbolic spinor representation of a spacelike curve defined on an oriented (spacelike or time-like) surface in Minkowski space  $\mathbb{R}_1^3$ . Then, we obtain the relation between the hyperbolic spinor representation of the Frenet frame of the spacelike curve on oriented surface and Darboux frame of the surface on the same points. Finally, we give one example about these hyperbolic spinors.

### 1. Introduction

Spinors used to expand space vector concepts in orthogonal group theory in particular such as rotation or Lorentzian groups in mathematics and physics are the elements of the complex vector space. For the first time, in the geometrical meaning, spinors has been studied by the French mathematician E. Cartan. Cartan obtained these spinors consisting of two complex components in terms of vectors in three-dimensional Euclidean space, [3].

The word “spinor” was coined by P. Ehrenfest in his work on quantum physics, [19]. Moreover, W. Pauli introduced spin matrices and first applied spinors to mathematical physics, [17]. Then P. A. M. Dirac showed the connection between spinors and the Lorentz groups. Moreover, Dirac discovered the fully relativistic theory of electron spin, [7]. The other uses of the spinors are also available. For example, in the mechanics of solids, the components of unit spinors have been used for over 50 years as “Cayley-Klein parameters”. Also spinor theory is also associated with electricity transmission line, [6].

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Two-component spinors algebra and the Pauli matrices which constitute a part of special unitary group allows to give the more smooth and nice definition of three-dimensional real space rotation rather than the classic definition. Also, spinor transformations leave the same physical image since operators of quantum mechanics and dynamic variables of classical mechanics are similar, [14].

For the first time, the applications in astronomy mechanics problems of spinors were made by P. Kustaanheimo and E. Stiefel, [13]. They originally spinors are used by atomic physicists and mathematicians, substantially. Recently they have began to be studied by the astronomers.

Castillo and Barrales published a study about spinors in three-dimensional real space  $\mathbb{R}^3$ . In their study, the Frenet triad along the curve in 3-dimensional real space have been corresponded to a single vector which has two complex components called spinor, [5]. In another study, Castillo presented spinor formulation in the sense that elementary in four-dimensional space, [4]. Ünal et. al., studied the spinor Bishop equations in three-dimensional Euclidean space  $\mathbb{R}^3$ . Moreover, they gave the relation between the spinor formulation of Bishop and Frenet frames, [20]. Similarly, Kisi and Tosun expressed the spinor formulation of Darboux frame on the oriented surface in 3-dimensional Euclidean space, [12]. Then, Ketenci et. al. obtained that the hyperbolic spinor formulation of a non-null regular curve in Minkowski 3-space, [11]. In addition that, Erisir et. al. expressed the rotation, element of  $SO(1, 3)$ , between the Frenet frame and the other frame defined as alternatively of the (spacelike or timelike) curves in Minkowski space  $\mathbb{R}_1^3$  in terms of the rotation, element of  $SU(2, H)$ , with the aid of the hyperbolic spinors, [9].

The purpose of this study, Darboux frame of a (spacelike or timelike) surface in Minkowski space  $\mathbb{R}_1^3$  is symbolized by the aid of spinors with two hyperbolic components. Also the hyperbolic spinor representation of the relation between Frenet frame of the spacelike curve taken on the oriented surface in  $\mathbb{R}_1^3$  and Darboux frame of the surface at the same point in Minkowski 3-space is obtained. Finally, an example of supporting theorems and results of this study is obtained.

## 2. Preliminaries

In 3-dimensional Minkowski space, for  $\mathbf{u} = (u_1, u_2, u_3)$ ,  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}_1^3$  the standard metric is introduced by

$$(2.1) \quad g(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 - u_3v_3.$$

For an arbitrary vector  $\mathbf{u}$  in  $\mathbb{R}_1^3$ , if we take  $g(\mathbf{u}, \mathbf{u}) > 0$  (or  $\mathbf{u} = \mathbf{0}$ ),  $g(\mathbf{u}, \mathbf{u}) = 0$  ( $\mathbf{u} \neq \mathbf{0}$ ) or  $g(\mathbf{u}, \mathbf{u}) < 0$ , this vector is called spacelike, lightlike (null) or timelike vector, respectively. Moreover, if the first component of the timelike vector  $\mathbf{u} \in \mathbb{R}_1^3$  is positive or negative, then this vector is called future pointing or past pointing timelike vector, respectively.

The norm of the vector  $\mathbf{u} \in \mathbb{R}_1^3$  is introduced by

$$(2.2) \quad \|\mathbf{u}\|_L = \sqrt{|g(\mathbf{u}, \mathbf{u})|}.$$

Also, the Lorentzian vector product is

$$(2.3) \quad \mathbf{u} \wedge_L \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

[15].

Let  $\alpha$  be an arbitrary curve in Minkowski space  $\mathbb{R}_1^3$ . If the velocity vector  $\alpha'$  of the curve  $\alpha$  is spacelike, lightlike (null) or timelike vector, then, this curve is called spacelike, lightlike (null) or timelike curve. Moreover, let  $s$  be arc-length parameter of the curve  $\alpha$ . So,  $\|\alpha'(s)\|_L = 1$  and the curve  $\alpha$  has unit speed, [15]. Throughout this work, unless otherwise stated the curves will be arc-parameterized  $s$  and regular.

Let the curve  $\alpha : I \rightarrow \mathbb{R}_1^3$  be the spacelike curve and the tangent, normal and binormal vector of this curve be  $\mathbf{T}(s)$ ,  $\mathbf{N}(s)$  and  $\mathbf{B}(s)$ . So, these vectors are calculated by

$$(2.4) \quad \mathbf{T}(s) = \alpha'(s), \quad \mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|_L}, \quad \mathbf{B}(s) = \varepsilon_B (\mathbf{T}(s) \wedge_L \mathbf{N}(s)).$$

Thus, the vector fields  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  of the spacelike curve  $\alpha$  in  $\mathbb{R}_1^3$  called Serret-Frenet frame. The Frenet formulas are

$$(2.5) \quad \begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= \varepsilon_B \kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= \tau \mathbf{N} \end{aligned}$$

or

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \varepsilon_B \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

where  $\varepsilon_B = \langle \mathbf{B}, \mathbf{B} \rangle_L$  and  $\kappa, \tau$  are the curvature and torsion functions of the curve  $\alpha$ , [10].

Let  $M$  be a surface in Minkowski space  $\mathbb{R}_1^3$ . If the induced metric on the surface is a Lorentzian metric, then this surface is called a timelike surface. Also, the surface  $M$  is called a spacelike surface if the induced metric on this surface is a positive definite Riemannian metric. Moreover, the normal vector of the spacelike (timelike) surface is a timelike (spacelike) vector, [8].

Now, let  $M$  in  $\mathbb{R}_1^3$  be oriented (spacelike or timelike) surface. We consider that the tangent vector of an arbitrary spacelike curve  $\alpha$  given on the surface  $M$  is  $\mathbf{T}(s)$ . Then, we take the oriented unit normal vector  $\mathbf{n}(s)$  at the this point  $\alpha(s)$ . Thus, the vector  $\mathbf{g}(s) = \varepsilon_{\mathbf{g}}(\mathbf{T}(s) \wedge_L \mathbf{n}(s))$  can be defined. From here, the triad  $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$  is called by the **Darboux frame** of this (spacelike or timelike) surface. Moreover the rotation matrix between Frenet frame and Darboux frame is

$$(2.6) \quad \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{pmatrix}$$

where  $\theta$  is hyperbolic angle, [16]. Under this transformation, timelike vectors turn into timelike vectors; spacelike vectors turn into spacelike vectors.

Moreover, for the spacelike curve, considering the derivative of the equations in the matrix (2.6), the Darboux formulas are equal to

$$(2.7) \quad \begin{aligned} \mathbf{T}' &= \kappa_n \mathbf{n} + \kappa_g \mathbf{g} \\ \mathbf{n}' &= \varepsilon_{\mathbf{B}} \kappa_n \mathbf{T} + \tau_g \mathbf{g} \\ \mathbf{g}' &= -\varepsilon_{\mathbf{B}} \kappa_g \mathbf{T} + \tau_g \mathbf{n} \end{aligned}$$

or

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{g}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & \kappa_g \\ \varepsilon_{\mathbf{B}} \kappa_n & 0 & \tau_g \\ -\varepsilon_{\mathbf{B}} \kappa_g & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{g} \end{pmatrix}$$

where  $\kappa_n = \kappa \cosh \theta$ ,  $\kappa_g = \kappa \sinh \theta$  and  $\tau_g = \tau - \frac{d\theta}{ds}$ , [16].

### 3. The hyperbolic spinors representation for the spacelike curve

We consider that numbers of the form  $z = x + jy$  where  $x, y \in \mathbb{R}$  and  $j^2 = 1$ . This number system is known as the "hyperbolic number system" symbolized

$$\mathbb{H} = \{z = x + jy : x, y \in \mathbb{R}, j^2 = 1, j \neq \mp 1\}.$$

Moreover, the hyperbolic number  $\bar{z} = x - jy$  is called the hyperbolic conjugate of  $z$ . For  $x, y, u, v \in \mathbb{R}$  and  $z = x + jy$ ,  $w = u + jv$  the inner product and modulus squatted in  $\mathbb{H}$  are

$$\langle z, w \rangle = \text{Re}(z\bar{w}) = \text{Re}(\bar{z}w) = xu - yv, \quad |z|^2 = \bar{z}z = x^2 - y^2$$

respectively. On the other hand for the hyperbolic numbers, the equations  $\overline{z + w} = \bar{z} + \bar{w}$ ,  $\overline{\bar{z}.w} = \bar{z}.w$  and  $\overline{\bar{z}} = z$  are hold. Moreover, the Euler formula is  $e^{j\theta} = \cosh \theta + j \sinh \theta$  for  $\theta \in \mathbb{R}$ , [21].

Let us suppose  $A$  be an  $n \times n$  Hermitian matrix ( $\overline{A^t} = A$ ) defined on  $\mathbb{H}$  and  $U = e^{jA}$ . So, the set of all matrices  $n \times n$  on  $\mathbb{H}$  satisfying  $\overline{U^t}U = U\overline{U^t} = 1$  forms a group. This group is called the hyperbolic unitary group and denoted as  $U(n, \mathbb{H})$ . Especially, if we take  $\det U = 1$ , this special group is denoted by  $SU(n, \mathbb{H})$ , [21].

The Lorentz group is defined as the group of all Lorentz transformations of Minkowski space. Also, Lorentz transformations which preserve the direction of time are called orthochronous. The subgroup of orthochronous transformations is often denoted  $O^+(1, 3)$ . They have determinant +1 and they preserve the orientation. This subgroup is denoted by  $SO(1, 3)$ , [2].

There is a homomorphism between the group of rotation about the origin;  $SO(1, 3)$  and the group of unitary matrix in the type of  $2 \times 2$ ;  $SU(2, \mathbb{H})$ . While the elements of  $SO(1, 3)$  activate the vectors with three real component in Minkowski space  $\mathbb{R}_1^3$ , the elements of  $SU(2, \mathbb{H})$  activate the hyperbolic spinors, [18].

A hyperbolic spinor is introduced as

$$(3.1) \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where  $\psi_1, \psi_2 \in \mathbb{H}$ . For  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_1^3$ , if we consider that  $\mathbf{a} + j\mathbf{b} = (x_1, x_2, x_3)$  is an isotropic (null) vector then, we can write that  $x_1^2 + x_2^2 - x_3^2 = 0$ ,  $\mathbf{a} + j\mathbf{b} \neq \mathbf{0}$ .

Moreover, each vector in  $\mathbb{R}_1^3$  corresponds to the matrix

$$X = \begin{pmatrix} x_3 & x_1 - jx_2 \\ x_1 + jx_2 & -x_3 \end{pmatrix}.$$

So, using the Pauli matrices  $P$  corresponding to the standard basis of  $\mathbb{R}_1^3$ , the vector  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  consisting of hyperbolic symmetric matrices is

$$(3.2) \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

PROPOSITION 3.1. For two arbitrary hyperbolic spinors  $\varphi$  and  $\psi$ , the following statements are hold;

$$\begin{aligned} i) \quad \widehat{\psi} &= - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\psi} = \begin{pmatrix} -\overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix} \\ ii) \quad \widehat{\widehat{\psi}} &= -\psi \\ iii) \quad \varphi^t \sigma \psi &= \psi^t \sigma \varphi, \end{aligned}$$

[11].

For  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_1^3$

$$(3.3) \quad \begin{aligned} \mathbf{a} + j\mathbf{b} &= \psi^t \sigma \psi = (\psi_1^2 - \psi_2^2, j(\psi_1^2 + \psi_2^2), -2\psi_1\psi_2) \\ \mathbf{c} &= -\widehat{\psi}^t \sigma \psi = (\psi_1\overline{\psi_2} + \overline{\psi_1}\psi_2, j(\psi_1\overline{\psi_2} - \psi_1\overline{\psi_2}), |\psi_1|^2 - |\psi_2|^2) \end{aligned}$$

where the mutually orthogonal vectors and  $\|\mathbf{a}\|_L = \|\mathbf{b}\|_L = \|\mathbf{c}\|_L = \overline{\psi}^t \psi$ . Also, the vector  $\mathbf{c} \in \mathbb{R}_1^3$  can be obtained as  $\mathbf{c} = \overline{\psi}^t P \psi$ . On the other hand, there is a hyperbolic spinor providing the equation (3.3) for mutually orthogonal vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_1^3$  which have same magnitude, [11].

For an arbitrary matrix  $U \in SU(2, \mathbb{H})$ , the equation  $\overline{\psi'}^t \psi' = \overline{\psi}^t \psi$  is hold where  $\psi' = U\psi$ . So, the magnitudes of the spacelike (or timelike) vectors  $\mathbf{a}', \mathbf{b}', \mathbf{c}'$  be connected with the hyperbolic spinor  $\psi'$  are equal to the magnitudes of the spacelike (or timelike) vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be connected with the hyperbolic spinor  $\psi$ . Thus, each element of  $SU(2, \mathbb{H})$  creates a transformation that converts the orthogonal basis  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  of  $\mathbb{R}_1^3$  to the orthogonal basis  $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ . Moreover, two elements of  $SU(2, \mathbb{H})$ ;  $U$  and  $-U$  generate the same ordered triad of  $\mathbb{R}_1^3$ . While the triad  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  corresponds to the hyperbolic spinor  $\psi$ , the ordered triads  $\{\mathbf{b}, \mathbf{c}, \mathbf{a}\}$  and  $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$  correspond to different hyperbolic spinors. In addition, the spinors  $\psi$  and  $-\psi$  correspond to the same triad.

Let the hyperbolic spinor  $\psi$  be non-zero, then we consider that

$$\det(\psi, \widehat{\psi}) = \begin{vmatrix} \psi_1 & -\overline{\psi_2} \\ \psi_2 & \overline{\psi_1} \end{vmatrix} = |\psi_1|^2 + |\psi_2|^2.$$

This sum solely is zero when both  $\psi_1$  and  $\psi_2$  hyperbolic components are zero. So,  $\{\psi, \widehat{\psi}\}$  is linear independent. Moreover,  $\psi_1$  and  $\psi_2$  create a basis for the spinors with two hyperbolic components, [11].

The Frenet frame of the spacelike curve  $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$  corresponds to a hyperbolic spinor  $\psi$ . From the equation (3.3), it can be written

$$\mathbf{N} + j\mathbf{B} = \psi^t \sigma \psi, \quad \mathbf{T} = -\widehat{\psi}^t \sigma \psi.$$

So, the following theorem can be given.

**THEOREM 3.2.** *If the hyperbolic spinor  $\psi$  represents the oriented triad  $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$  of the spacelike curve  $\alpha$  in Minkowski space  $\mathbb{R}_1^3$ , the Frenet derivative equations are equivalent a single hyperbolic spinor equation as*

$$\frac{d\psi}{ds} = \frac{1}{2} (j\tau\psi - \varepsilon_B \kappa \hat{\psi}),$$

[11].

#### 4. The hyperbolic spinor Darboux frame for the spacelike curves

In this section, we give that the relation between hyperbolic spinor representation of Darboux frame on the oriented surface (spacelike or timelike) and the Frenet frame of the spacelike curve on this surface. Then, we give one example about these hyperbolic spinors.

Let  $M$  in  $\mathbb{R}_1^3$  be the oriented (spacelike or timelike) surface and  $\alpha$  be the unit speed spacelike curve on the surface  $M$ . According to the results in the previous section, we know that the each triad of  $\mathbb{R}_1^3$  corresponds to the one hyperbolic spinor. So, we can take that the Darboux triad  $\{\mathbf{n}, \mathbf{g}, \mathbf{T}\}$  corresponds to the hyperbolic spinor  $\Gamma$ . Thus, we can write

$$(4.1) \quad \mathbf{n} + j\mathbf{g} = \Gamma^t \sigma \Gamma, \quad \mathbf{T} = -\hat{\Gamma}^t \sigma \Gamma$$

where  $\bar{\Gamma}^t \Gamma = 1$ .

Thus, while the hyperbolic spinor  $\Gamma$  represent the Darboux triad  $\{\mathbf{n}, \mathbf{g}, \mathbf{T}\}$ ,  $\frac{d\Gamma}{ds}$  represent the change of this triad during the spacelike curve. Since the binary  $\{\Gamma, \hat{\Gamma}\}$  create a basis for the hyperbolic spinors, the statement, we can write

$$(4.2) \quad \frac{d\Gamma}{ds} = h\Gamma + k\hat{\Gamma}.$$

Differentiating the first equation in (4.1), we find easily that

$$(4.3) \quad \frac{d\mathbf{n}}{ds} + j \frac{d\mathbf{g}}{ds} = \frac{d\Gamma^t}{ds} \sigma \Gamma + \Gamma^t \sigma \frac{d\Gamma}{ds}.$$

So, considering the equations (2.5), (3.2) and (4.3), we can write

$$\begin{aligned} (\varepsilon_B \kappa_n \mathbf{T} + \tau_g \mathbf{g}) + j(-\varepsilon_B \kappa_g \mathbf{T} + \tau_g \mathbf{n}) &= (h\Gamma + k\hat{\Gamma})^t \sigma \Gamma + \Gamma^t \sigma (h\Gamma + k\hat{\Gamma}) \\ &= h(\Gamma^t \sigma \Gamma + \Gamma^t \sigma \Gamma) + k(\hat{\Gamma}^t \sigma \Gamma + \Gamma^t \sigma \hat{\Gamma}). \end{aligned}$$

Thus, the equation

$$\varepsilon_B (\kappa_n - j\kappa_g) \mathbf{T} + j\tau_g (\mathbf{n} + j\mathbf{g}) = 2h (\mathbf{n} + j\mathbf{g}) - 2k (\mathbf{T})$$

is hold. Finally, we can see easily that

$$(4.4) \quad h = \frac{j\tau_g}{2}, \quad k = \frac{\varepsilon_B (-\kappa_n + j\kappa_g)}{2}.$$

So, from the equations (4.2) and (4.4), the following theorem can be given.

**THEOREM 4.1.** *The hyperbolic spinor  $\Gamma$  corresponds to the triad  $\{\mathbf{n}, \mathbf{g}, \mathbf{T}\}$  on the oriented (spacelike or timelike) surface  $M$  associated with spacelike curve. So, Darboux derivative equations have single hyperbolic spinor as follows*

$$\frac{d\Gamma}{ds} = \left( \frac{j\tau_g}{2} \right) \Gamma + \varepsilon_B \left( \frac{-\kappa_n + j\kappa_g}{2} \right) \hat{\Gamma}$$

where  $\tau_g$ ,  $\kappa_g$  and  $\kappa_n$  are geodesic torsion, geodesic curvature and normal curvature, respectively.

Now, from the equations (3.4) and (4.1) we give that the relation between the hyperbolic spinor corresponding to the frames  $\{\mathbf{n}, \mathbf{g}, \mathbf{T}\}$  and  $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$  in Minkowski space  $\mathbb{R}_1^3$ . So, if we consider that the equation (2.4)

$$\begin{aligned} \psi^t \sigma \psi &= \mathbf{N} + j\mathbf{B} = \mathbf{n} \cosh \theta + \mathbf{g} \sinh \theta + j\mathbf{n} \sinh \theta + j\mathbf{g} \cosh \theta \\ &= \mathbf{n} (\cosh \theta + j \sinh \theta) + j\mathbf{g} (\cosh \theta + j \sinh \theta) \\ &= (\cosh \theta + j \sinh \theta) (\mathbf{n} + j\mathbf{g}) \\ &= e^{j\theta} (\mathbf{n} + j\mathbf{g}) \\ &= e^{j\theta} (\Gamma^t \sigma \Gamma). \end{aligned}$$

So, we can give the following theorem.

**THEOREM 4.2.** *Let  $M$  be oriented (spacelike or timelike) surface in Minkowski space  $\mathbb{R}_1^3$  and  $\alpha$  be the spacelike curve on the surface  $M$ . The relation between the hyperbolic spinors  $\psi$  and  $\Gamma$  corresponding to the Frenet and Darboux frames  $\{\mathbf{n}, \mathbf{g}, \mathbf{T}\}$  and  $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ , respectively, is*

$$\psi^t \sigma \psi = e^{j\theta} \Gamma^t \sigma \Gamma$$

From here, we can give the example about this hyperbolic spinor.

**EXAMPLE 4.3.** *Let the curve  $\alpha : I \rightarrow \mathbb{R}_1^3$  be as  $\alpha(s) = (1, \sinh(s), \cosh(s))$ . This curve is unit speed and spacelike curve with timelike normal. The*



Frenet vector fields are calculated by

$$\begin{aligned}\mathbf{T} &= (0, \cosh(s), \sinh(s)) \\ \mathbf{N} &= (0, \sinh(s), \cosh(s)) \\ \mathbf{B} &= (1, 0, 0)\end{aligned}$$

where  $g(\mathbf{T}, \mathbf{T}) = 1$ ,  $g(\mathbf{N}, \mathbf{N}) = -1$ ,  $g(\mathbf{B}, \mathbf{B}) = 1$ . From here if we make necessary arrangements, we can see that  $\kappa = 1$ ,  $\tau = 0$ . Since  $\varepsilon_B = +1$  from the Theorem 3.1., we can find that

$$\frac{d\psi}{ds} = -\frac{1}{2}\hat{\psi}, \quad \frac{d\hat{\psi}}{ds} = \frac{1}{2}\psi$$

and

$$\psi = c_1 \cos\left(\frac{s}{2}\right) - c_2 \sin\left(\frac{s}{2}\right), \quad \hat{\psi} = c_1 \sin\left(\frac{s}{2}\right) + c_2 \cos\left(\frac{s}{2}\right)$$

where  $c_1, c_2 \in \mathbb{H}$ .

## 5. Conclusion

In geometry and physics, spinors can be defined as elements of complex vector space associated with the Euclidean space. Spinors for the first time was discovered by the French mathematician Elie Cartan, who also has a basic work about Lie groups, a modern theory. One of the main aims of Cartan in this study is to systematically develop theory of spinors giving the geometric definition of this mathematical statement and in addition to this, to make a contribution to differential geometry, group theory and mathematical physics. Spinors resemble geometric vectors or, more generally tensors. So, spinors make linear transformations when the Euclidean space is subject to an infinitesimal rotation. It is an important feature of spinors. Actually, spinors are elements of the representation space of spin group. On the other hand, spinors can be defined as elements of the vector space that carries a linear representation of the Clifford algebra which is an associative algebra.

In this study, we show that how can be these curves expressed in terms of hyperbolic spinors. So, we think that this study would be important to the sciences of mathematics, engineering and astronomy. There is no doubt that this study will add innovation to science.

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